Chapter Three

Discrete time Systems

1-1 Introduction

An important subset of discrete-time (DT) systems satisfies the linearity and time-invariance properties, such DT systems are referred to as linear, time-invariant, discrete-time (LTID) systems. In the time domain, an LTID system is modeled either with a linear, constant-coefficient difference equation or with its impulse response.

1-2 Block Diagram Representation of DT Systems

The graphical symbols for the unit delay, unit advance, gain, and sum operator are shown in Figure (1) and described as below:

a) The unit delay operator D [·] shifts the signal one unit to the right. It is also shown by the symbol z^{-1}. $D[x(n)] = x(n-1)$
b) The unit advance operator A [·] shifts the signal one unit to the left. It is also shown by the symbol z. $A[x(n)] = x(n+1)$
c) The gain operator G [·] multiplies all elements of the discrete signal by a factor G (positive or negative).
d) The sum operator adds two discrete signals.

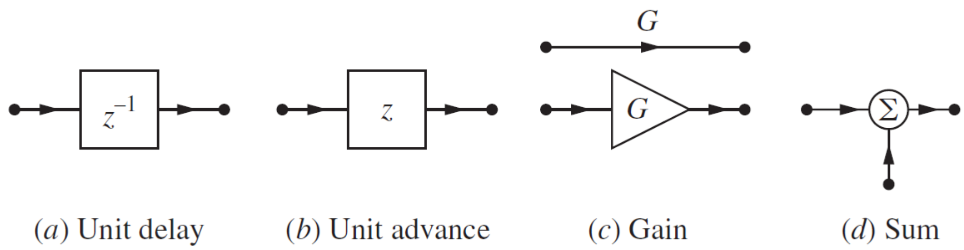


Figure 1: Block diagram units

Example 1: Draw the following: y[n] = x[n] +x [n-1] + x [n-2] + x [n-3]

Sol:

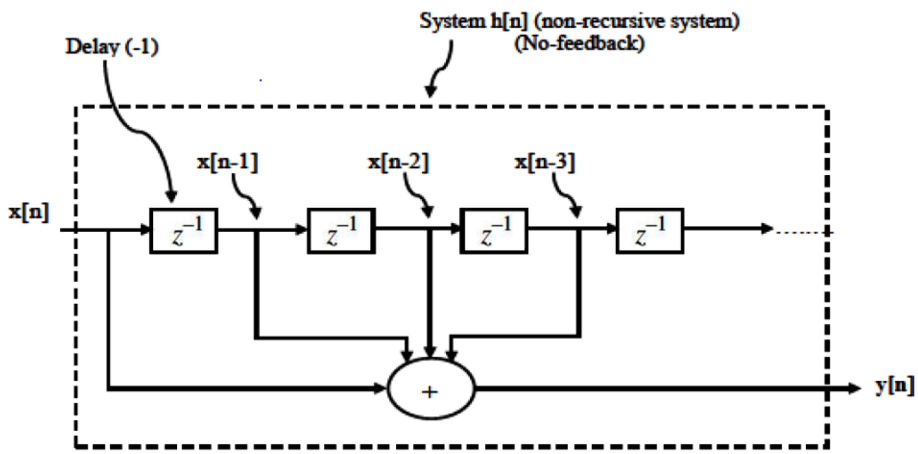


Figure 2: solution of example 1.

Example 2: Draw the following system: y[n] = y [n-1] - x[n]

Sol:

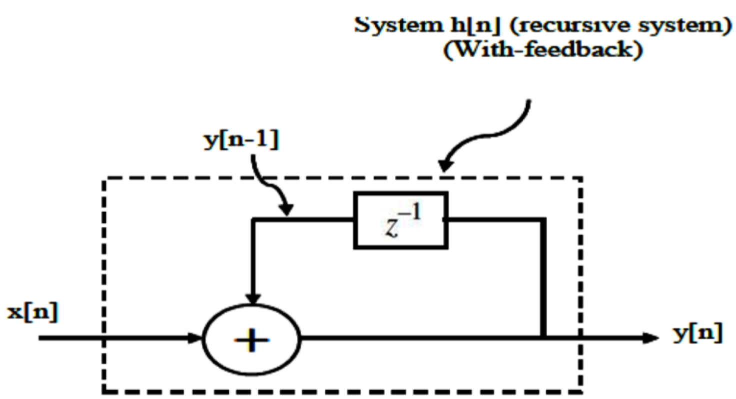


Figure 3: Ex. 2 solution.

Example 3: Draw the following system:

y[n] = x[n] – 1.9 x[n-1] + x[n-2] -0.9 y [n-2] +1.8 y [n-1]

Sol:

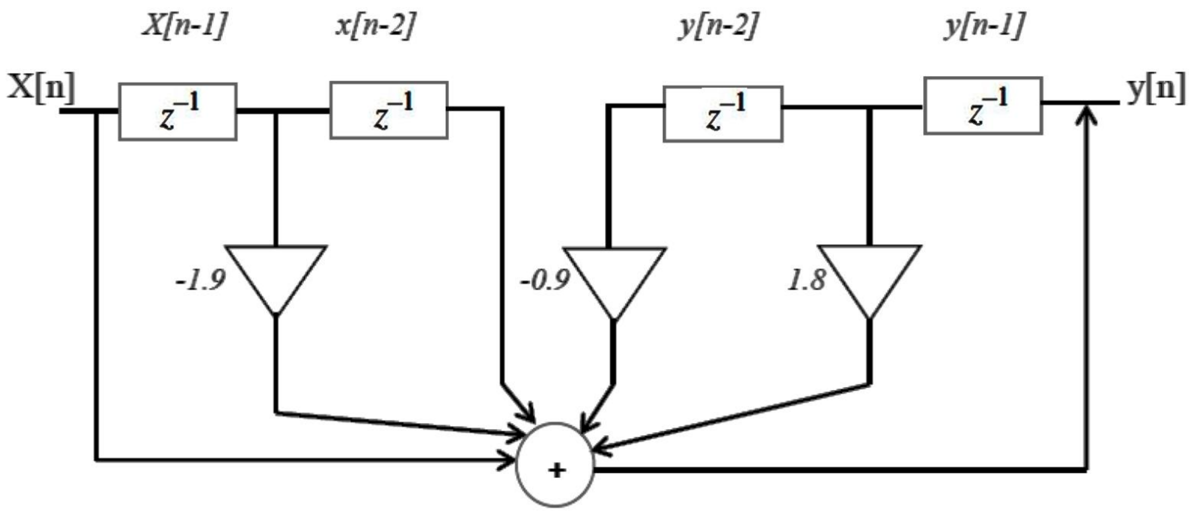


Figure 4: Ex. 3 solution.

Example 4: Write the equation of the following system:

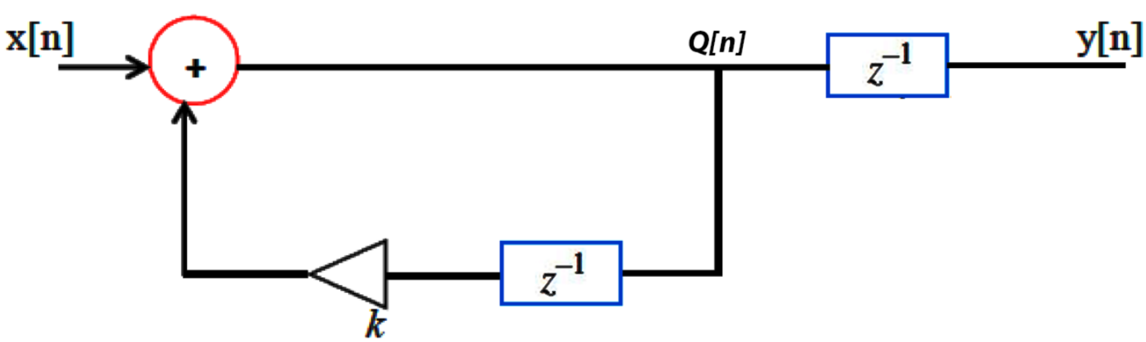


Figure 5: Ex. 4

Sol: Q[n] = x[n] +k Q [n-1] …….. Eq. (1)

y[n] = Q [n-1] ……………… Eq. (2)

by substituting Eq. (1) in Eq. (2) we get:

y[n] = x[n-1] + k Q[n-2] ………… Eq. (3) from Eq. (2) we can say: y[n-1] = Q [n-2]

Now the system equation is:

$y[n] = x[n-1] + k\, y[n-1]$

Example 5: Write the equation of the following system:

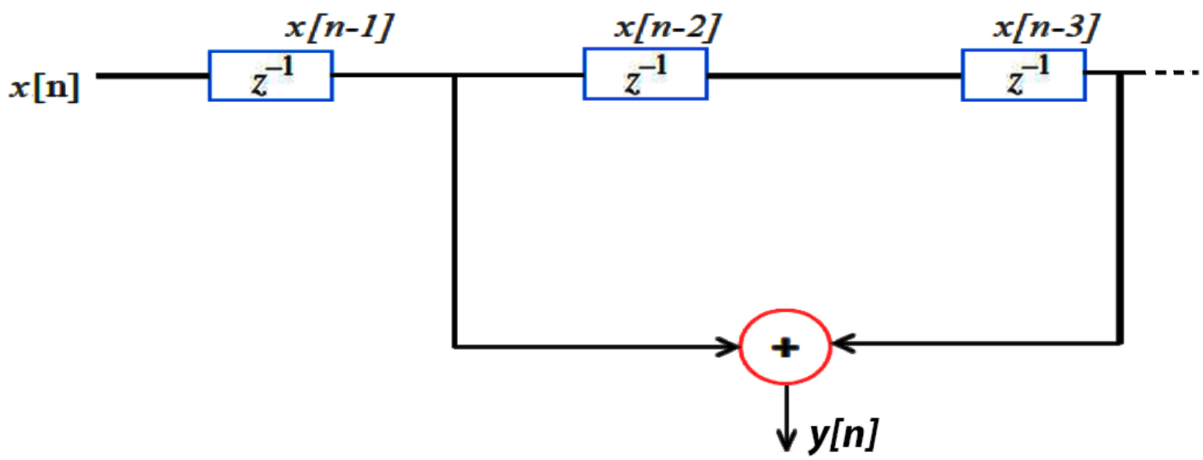


Figure 6: Ex. 5

Sol: $y[n] = x[n-1] + x[n-3]$

Example 6: Draw the system defined by the equation:

$y[n] = 0.5\, x[n] + 0.5\, x[n-1] + 0.8\, y[n-1]$

Sol:

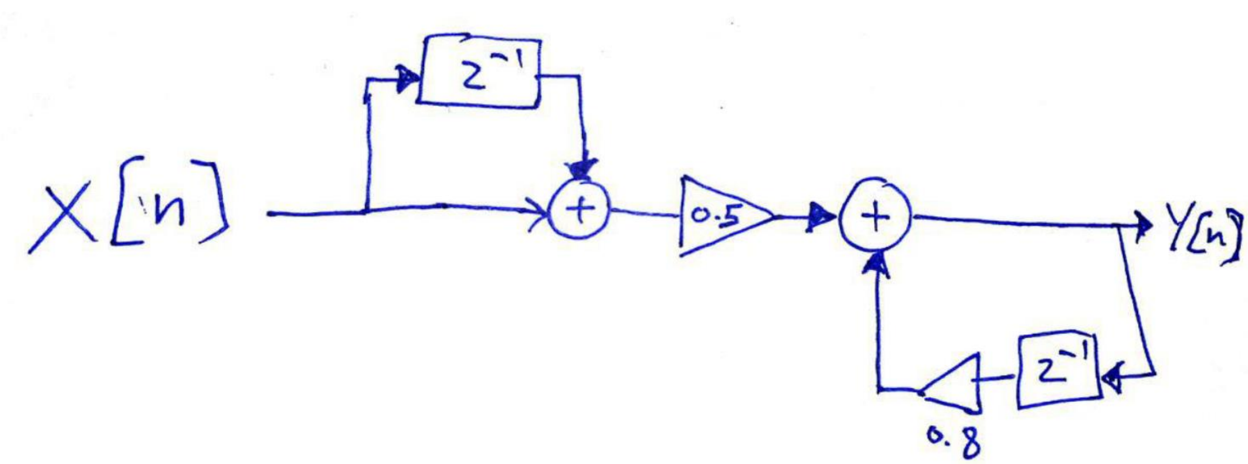
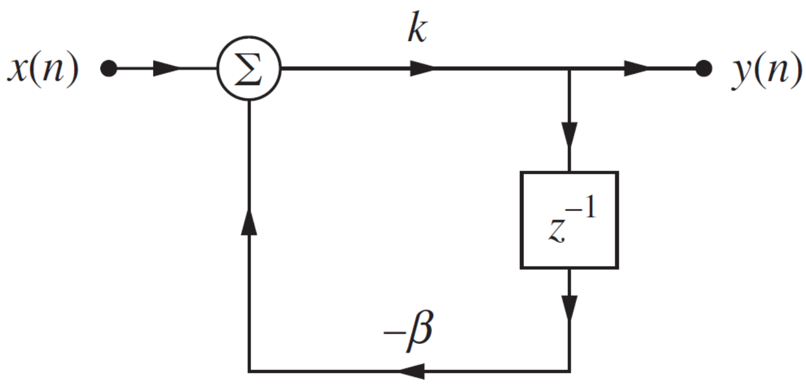


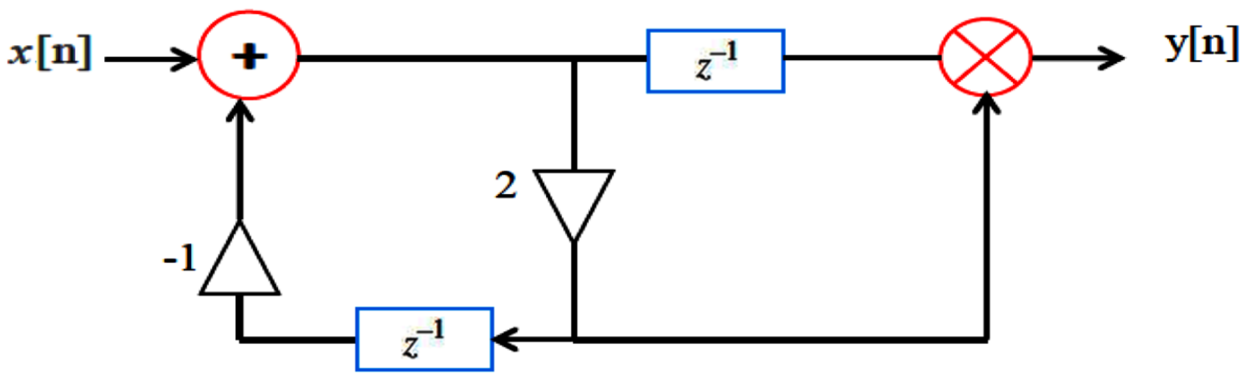
Figure 7: solution of Ex. 6

Example 7: write the equation of the block:

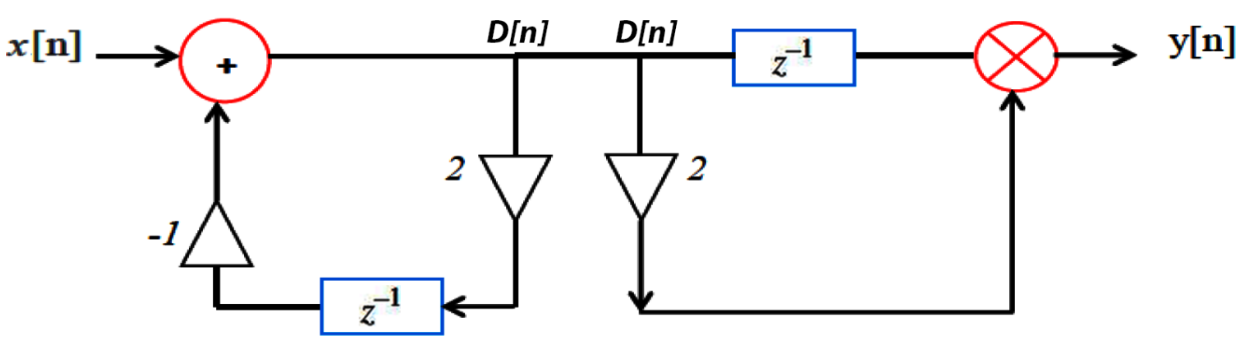


Sol: $y[n] = k\{ x[n] - \beta\, y[n-1] \}$

Example 8: Analyze the system and Write the equation of the following: للاطلاع



Sol: the block diagram could be redrawn as:



Now we can write the equations as

$D[n] = x[n] - 2\, D[n-1]$

D[n] + 2 D[n-1] = x[n] -----> D[n] $\{1+2z^{-1}\}$ = x [n]

So: D[n] = $\frac{x[n]}{1+2z^{-1}}$ …… Eq.1

y[n] = 2*D[n] * D[n-1] …………Eq.2

Now substituting Eq.1 to Eq.2 we get:

$$y[n] = 2 * \left\{\frac{x[n]}{1+2z^{-1}}\right\} * z^{-1}\left\{\frac{x[n]}{1+2z^{-1}}\right\}$$

$$= \left\{\frac{2 * x[n] * x[n-1]}{(1+2z^{-1})^2}\right\}$$

$$y[n] = \left\{\frac{2 * x[n] * x[n-1]}{1+4z^{-1}+4z^{-2}}\right\}$$

$$y[n]\{1+4z^{-1}+4z^{-2}\} = 2 * x[n] * x[n-1]$$

y[n] + 4 y[n-1] + 4 y[n-2] = 2 x[n] x[n-1]

y[n] = 2 x[n] x[n-1] - 4 y[n-1] - 4 y[n-2]

1-3 Difference Equation

Discrete system can be described using a difference equation, which takes the following form:

$$y(n) + a_1 y(n-1) + \ldots + a_N y(n-N) = b_0 x(n) + b_1 x(n-1) + \ldots + b_M x(n-M)$$

1-4 Types of System Responses

Discrete systems can be represented by the unit impulse response, unit step response, block diagram or difference equation.

- **Impulse Response**

A linear time- invariant system can be completely described by its unit-impulse response h(n), which is defined as the system response to the impulse input δ(n) with zero initial conditions as shown in the figure (11)

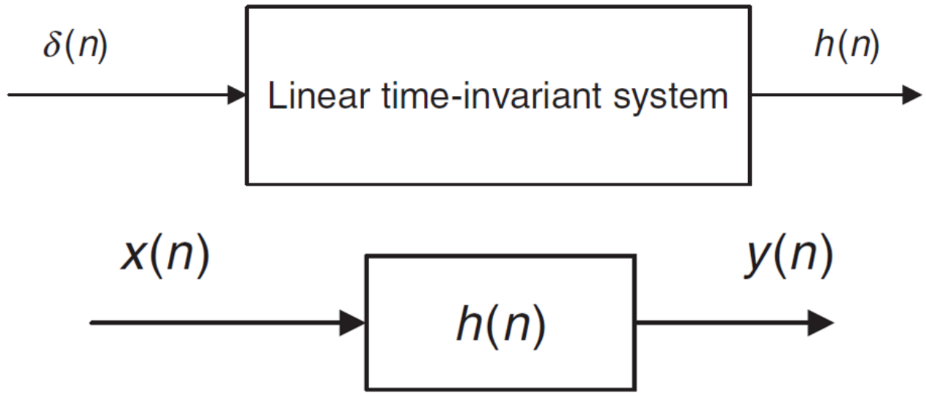


Figure 11: Unit impulse response

Example 9: Given the difference equation of the LTI system: $y(n) = 0.5x(n) + 0.25x(n-1)$

a. Determine the unit-impulse response $h(n)$.

b. Draw the system block diagram.

Solution:

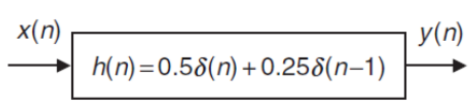
a. According to Figure 3.13, let $x(n) = \delta(n)$, then

$$h(n) = y(n) = 0.5x(n) + 0.25x(n-1) = 0.5\delta(n) + 0.25\delta(n-1).$$

Thus, for this particular linear system, we have

$$h(n) = \begin{cases} 0.5 & n = 0 \\ 0.25 & n = 1 \\ 0 & elsewhere \end{cases}$$

b. The block diagram of the linear time-invariant system is shown as



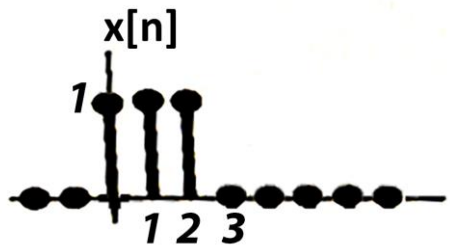
- **Unit Response**

The response of a LTI system to a unit step sequence U[n] is called the unit step response and denoted by S[n].

Example 9: Consider the LTI system:

y[n] = x[n] – 2 x[n-1] + 3 x[n-2]

What is the response to the sequence x[n] shown below:



Sol: **1-** directly we can substitute the values of x[n] and get the output as:

y[-1] = x[-1] – 2 x[-2] + 3x[-3] = 0 – 2*0 + 3*0 = 0

y[0] = x[0] – 2 x[-1] + 3x[-2] = 1 – 2*0 + 3*0 = 1

y[1] = 1 – 2*1 = -1

y[2] = 1 -2 +3 = 2

y[3] = x[3] – 2 x[2] + 3x[1] = 0 – 2*1 + 3*1 = 1

y[4] = 3

So now, y[n] = [<u>1</u> -1 2 1 3] where the underline points out the {n=0} instance

2- Using Convolution Sum equation

3- Using Convolution Sum Sliding Tape Method

4- Using Convolution Sum Array table

After the next section we will get back to this example where 2, 3 and 4 are a convolution between x[n] and h[n] which is h[n]=δ[n]-2 δ[n-1]+3 δ[n-2]

1-5 Convolution

The convolution of a LTI system can be expressed by the equation:

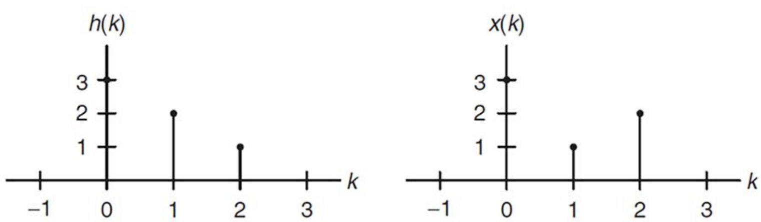
$$y(n) = \sum_{k=-\infty}^{\infty} x(k)h(n-k)$$
$$= \ldots + x(-1)h(n+1) + x(0)h(n) + x(1)h(n-1) + x(2)h(n-2) + \ldots$$

Example 10: Direct formula

Using the following sequences defined in Figure , evaluate the digital convolution

$$y(n) = \sum_{k=-\infty}^{\infty} x(k)h(n-k)$$

By applying the formula directly.



Solution:

Since we have samples from k=0 to k=2 , we can make the summation from 0 to 2 and rewrite the equation as follows:

$$y(n) = x(0)h(n) + x(1)h(n-1) + x(2)h(n-2)$$

$n = 0,\ y(0) = x(0)h(0) + x(1)h(-1) + x(2)h(-2) = 3 \times 3 + 1 \times 0 + 2 \times 0 = 9,$

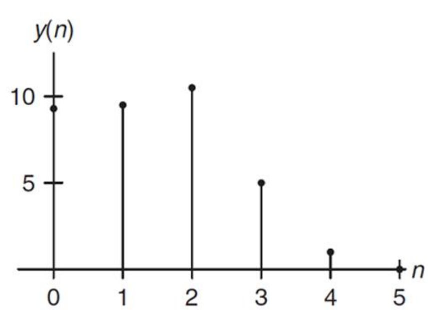
$n = 1,\ y(1) = x(0)h(1) + x(1)h(0) + x(2)h(-1) = 3 \times 2 + 1 \times 3 + 2 \times 0 = 9,$

$n = 2,\ y(2) = x(0)h(2) + x(1)h(1) + x(2)h(0) = 3 \times 1 + 1 \times 2 + 2 \times 3 = 11,$

$n = 3,\ y(3) = x(0)h(3) + x(1)h(2) + x(2)h(1) = 3 \times 0 + 1 \times 1 + 2 \times 2 = 5.$

$n = 4,\ y(4) = x(0)h(4) + x(1)h(3) + x(2)h(2) = 3 \times 0 + 1 \times 0 + 2 \times 1 = 2,$

$n \geq 5,\ y(n) = x(0)h(n) + x(1)h(n-1) + x(2)h(n-2) = 3 \times 0 + 1 \times 0 + 2 \times 0 = 0.$



Example 11: solve example 10 using **Sliding Method. Table**

TABLE Convolution sum using the table method.

k:	-2	-1	0	1	2	3	4	5	
$x(k)$:			**3**	**1**	**2**				
$h(-k)$:	1	2	3						$y(0) = 3 \times 3 = 9$
$h(1-k)$		1	2	3					$y(1) = 3 \times 2 + 1 \times 3 = 9$
$h(2-k)$			1	2	3				$y(2) = 3 \times 1 + 1 \times 2 + 2 \times 3 = 11$
$h(3-k)$				1	2	3			$y(3) = 1 \times 1 + 2 \times 2 = 5$
$h(4-k)$					1	2	3		$y(4) = 2 \times 1 = 2$
$h(5-k)$						1	2	3	$y(5) = 0$ (no overlap)

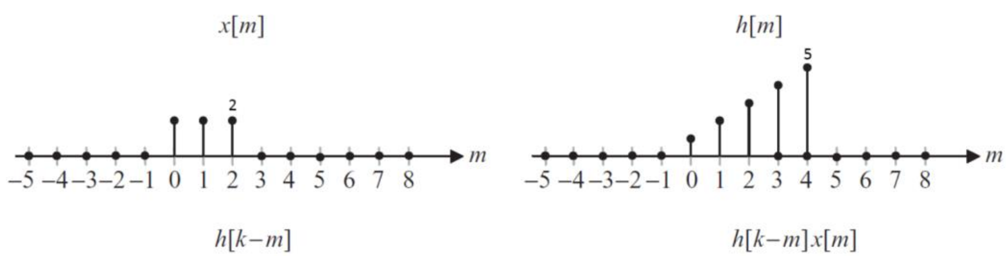
Example 12: Using Sliding Method. **Table**

For the following DT sequences:

$$x[k] = \begin{cases} 2 & 0 \le k \le 2 \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad h[k] = \begin{cases} k+1 & 0 \le k \le 4 \\ 0 & \text{otherwise,} \end{cases}$$

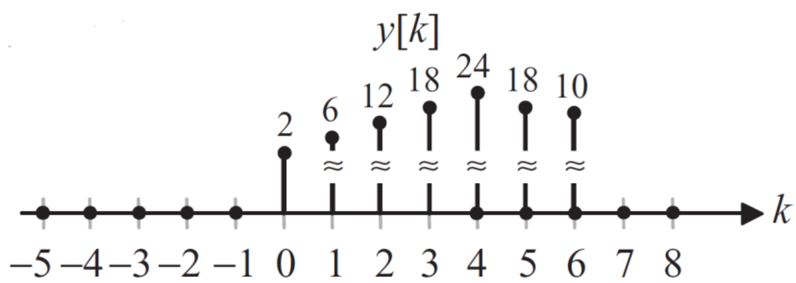
calculate the convolution sum $y[k] = x[k] * h[k]$

Solution:



m	...	-5	-4	-3	-2	-1	0	1	2	3	4	5	6	7	...	k	$y[k]$
$x[m]$							2	2	2								
$h[m]$							1	2	3	4	5						
$h[-m]$			5	4	3	2	1										
$h[-1-m]$		5	4	3	2	1										-1	0
$h[0-m]$			5	4	3	2	1									0	2
$h[1-m]$				5	4	3	2	1								1	6
$h[2-m]$					5	4	3	2	1							2	12
$h[3-m]$						5	4	3	2	1						3	18
$h[4-m]$							5	4	3	2	1					4	24
$h[5-m]$								5	4	3	2	1				5	18
$h[6-m]$									5	4	3	2	1			6	10
$h[7-m]$										5	4	3	2	1		7	0

Now we can draw the output sequence y[k]:

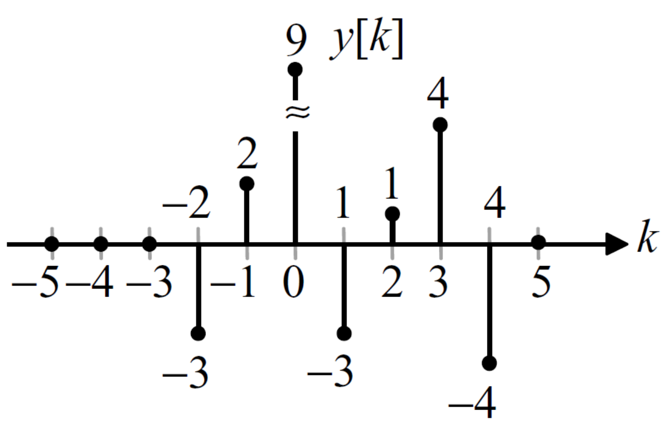


Example 12: using Sliding Tape Method

For the following pair of the input sequence $x[k]$ and impulse response $h[k]$:

$$x[k] = \begin{cases} -1 & k = -1 \\ 1 & k = 0 \\ 2 & k = 1 \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad h[k] = \begin{cases} 3 & k = -1, 2 \\ 1 & k = 0 \\ -2 & k = 1, 3 \\ 0 & \text{otherwise,} \end{cases}$$

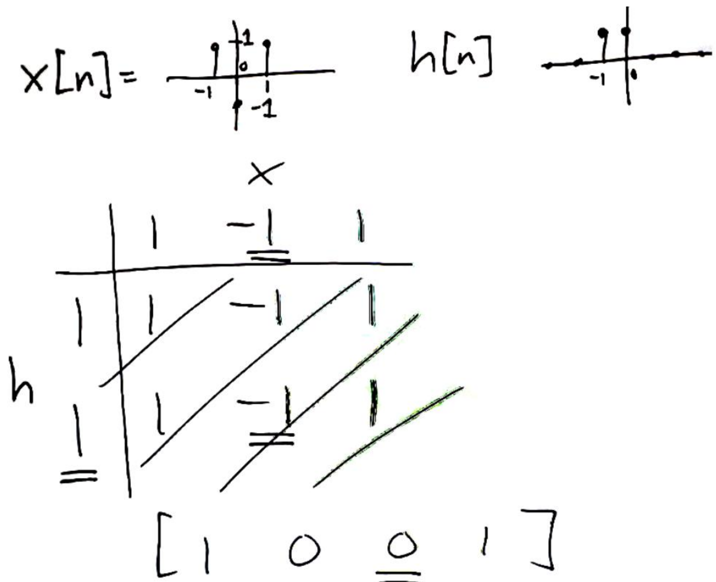
m	...	−5	−4	−3	−2	−1	0	1	2	3	4	5	6	...	k	$y[k]$
$h[m]$						3	1	−2	3	−2						
$x[m]$						−1	1	2								
$x[-m]$						2	1	−1								
$x[-3-m]$			2	1	−1										−3	0
$x[-2-m]$				2	1	−1									−2	−3
$x[-1-m]$					2	1	−1								−1	2
$x[0-m]$						2	1	−1							0	9
$x[1-m]$							2	1	−1						1	−3
$x[2-m]$								2	1	−1					2	1
$x[3-m]$									2	1	−1				3	4
$x[4-m]$										2	1	−1			4	−4
$x[5-m]$											2	1	−1		5	0



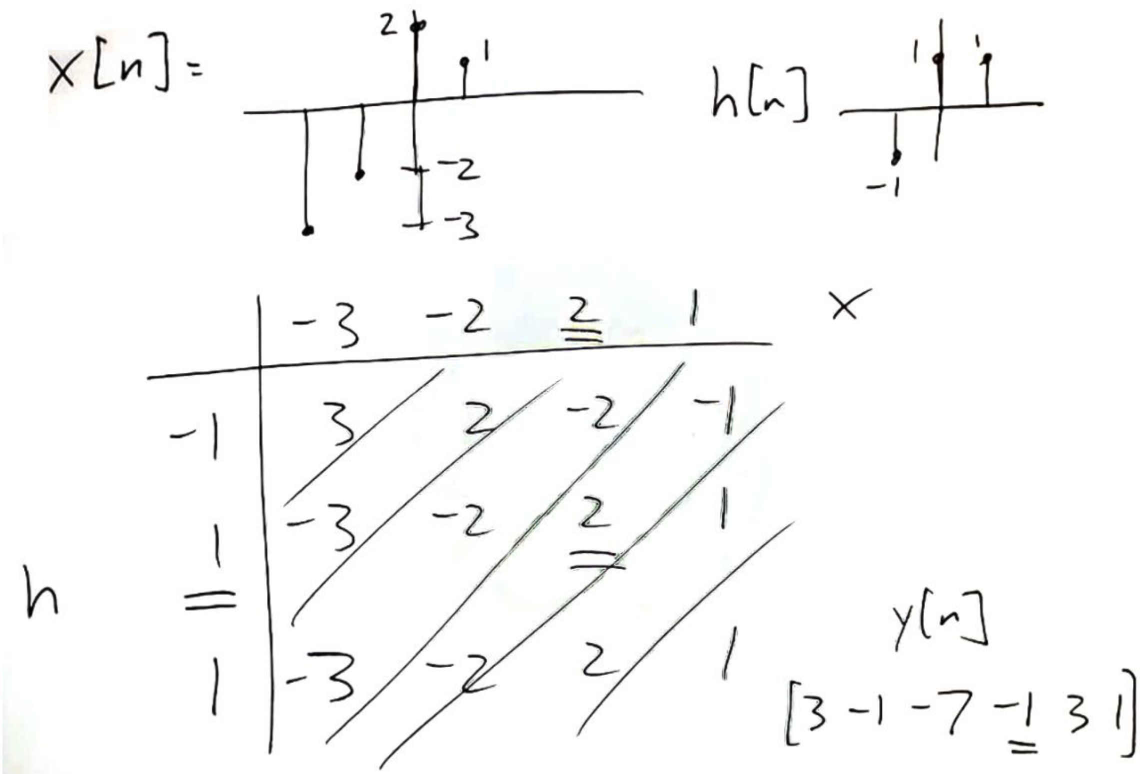
Example 12: using **Array Table** calculate the convolution of

x[n] = [1 -1 1] and h[n] = [1 1] where the double underline means n=0.

Solution:



Example 13:



Properties of the Convolution Sum

1. $x[n]*h[n]=h[n]*x[n]$ (Commutative)
2. $f[n]*\left[x[n]+y[n]\right]=f[n]*x[n]+f[n]*y[n]$ (Distributive)
3. $f[n]*\left[x[n]*y[n]\right]=\left[f[n]*x[n]\right]*y[n]$ (Associative)
4. $x[n-m]*h[n-k]=y[n-m-k]$ (Shifting)
5. $x[n]*\delta[n]=x[n]$

1-6 Correlation

The word co-relation means comparison. Basically it is the comparison between two signals or sequences. A good application of correlation is the radar system. In case of radar system the signal is transmitted from radar to the target and reflected back, the comparison will be between the transmitted and received signals. If there is similarity then target is detected.

Applications of correlation

The word co-relation means comparison. Basically it is the comparison

- Detection of difference between two signals.
- Identification of signals in the presence of noise
- Observing effects of inputs and outputs in control engineering

There are two types of Correlation: **Auto** and **Cross** Correlation

- Auto Correlation is between the signal and itself. As expressed by:

$$r_{xx}(l) = \sum_{n=-\infty}^{\infty} x(n)x(n-l)$$

- Cross Correlation is between two signals. As:

$$r_{xy}(l) = \sum_{n=-\infty}^{\infty} x(n)y(n-l)$$

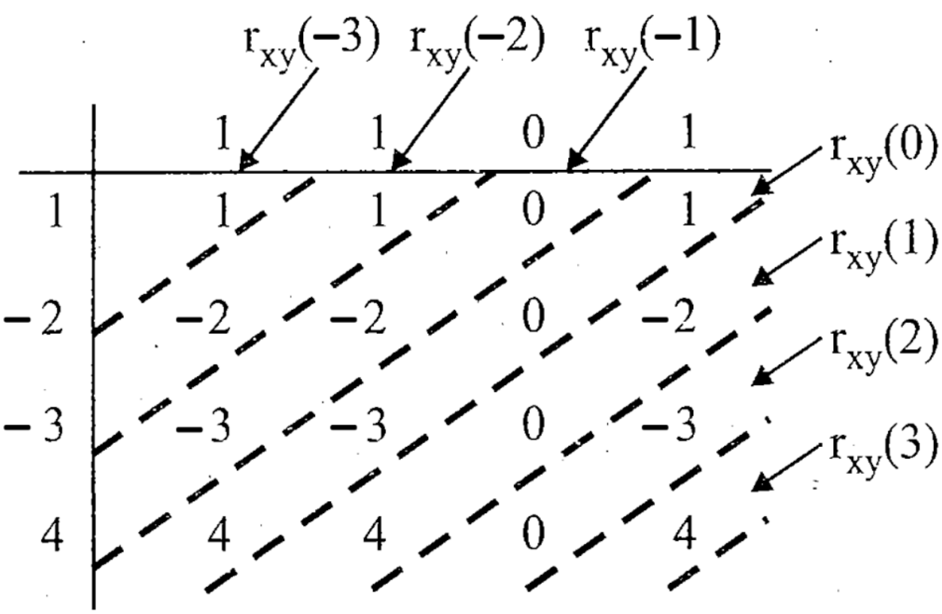
These two equations can be used directly to calculate the Correlation but there is a simple way to do that via the Table Method similar to the one used in the Convolution.

Example 14: find the Cross correlation r_{xy} of the signals:

$x(n) = \{1, 1, \underset{\uparrow}{0}, 1\}$ and $y(n) = \{4, -3, \underset{\uparrow}{-2}, 1\}$

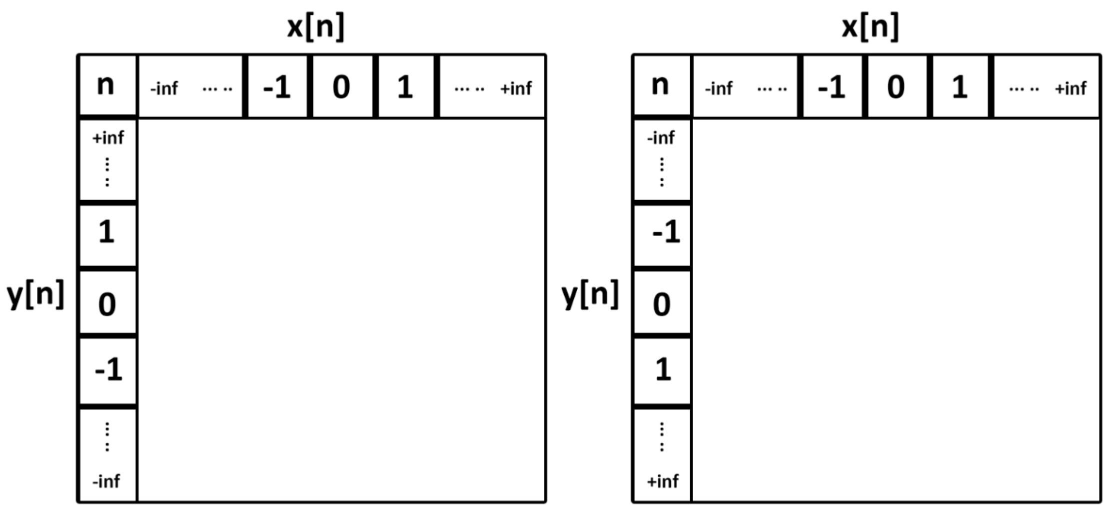
Solution:

			x[n]		↓n=0	
	n		1	1	0	1
	1	1	(1×1)	(1×1)	(1×0)	(1×1)
y[n]	**0**	-2	(-2×1)	(-2×1)	(-2×0)	(-2×1)
	-1	-3	(-3×1)	(-3×1)	(-3×0)	(-3×1)
	-2	4	(4×1)	(4×1)	(4×0)	(4×1)



$r_{xy}[n] = [1\ \ -1\ \ -5\ \ \underline{2}\ \ 2\ \ -3\ \ 4\,]$ the " $\underline{2}$ " is at n=0

Note: In **correlation** put positive index {n} of y[n] up, as seen n=1 is up. But in **convolution** as in example 13 y[-1] is up, negative index up. This the only difference between them in the Table method as illustrated here in the figure below:



Correlation **Convolution**

Example 15:

Determine the auto-correlation of the following signal :

(i) x (n) = { 1, 2, 1, 1}
↑

		x(n), n=0: 1	2	1	1
x(n)	1	(1×1)	(1×2)	(1×1)	(1×1)
	1	(1×1)	(1×2)	(1×1)	(1×1)
	2	(2×1)	(2×2)	(2×1)	(2×1)
n=0	1	(1×1)	(1×2)	(1×1)	(1×1)

$r_x[n]$ = [1 3 5 <u>7</u> 5 3 1] the " <u>7</u> " is at n=0