INVERSE z-TRANSFORM

Basically there are two methods to find the inverse of the z-transform:

1. Long Division Expansion
2. Partial Fractions Expansion

Partial Fractions Expansion

This method requires the knowledge of the z-transform of the basic sequences as shown in the table below:

Line No.	$x(n)$, $n\geq 0$	z-Transform $X(z)$	Region of Convergence
1	$x(n)$	$\sum_{n=0}^{\infty} x(n)z^{-n}$	
2	$\delta(n)$	1	$\|z\| > 0$
3	$au(n)$	$\frac{az}{z-1}$	$\|z\| > 1$
4	$nu(n)$	$\frac{z}{(z-1)^2}$	$\|z\| > 1$
5	$n^2u(n)$	$\frac{z(z+1)}{(z-1)^3}$	$\|z\| > 1$
6	$a^n u(n)$	$\frac{z}{z-a}$	$\|z\| > \|a\|$
7	$e^{-na}u(n)$	$\frac{z}{(z-e^{-a})}$	$\|z\| > e^{-a}$
8	$na^n u(n)$	$\frac{az}{(z-a)^2}$	$\|z\| > \|a\|$

Example 1:

Find the signal corresponding to

$$X(z) = \frac{4z}{(z+1)(z-0.8)}$$

Sol:

$$X(z) = \frac{4z}{(z+1)(z-0.8)} = \frac{A}{z+1} + \frac{B}{z-0.8}$$

$$\frac{X(z)}{z} = X_1(z) = \frac{4}{(z+1)(z-0.8)} = \frac{A}{z+1} + \frac{B}{z-0.8}$$

$$A = (z+1)X_1\Big|_{z=-1} = \frac{4}{z-0.8}\Big|_{z=-1} = \frac{4}{0.2} = 20$$

$$B = (z-0.8)X_1\Big|_{z=0.8} = \frac{4}{z+1}\Big|_{z=0.8} = \frac{4}{1.8} = 2.222$$

Thus,

$$X(z) = \frac{20z}{z+1} + \frac{2.22z}{z-0.8}$$

From Table 7.2, we obtain the inverse as

$$u[n] = 20(-1)^n + 2.22(0.8)^n$$

Example 2:

Find the inverse z-transform of the function

$$X(z) = \frac{z+1}{(z-0.5)(z-1)^2}$$

Solution

This example is on repeated poles. We let

$$X_1(z) = \frac{X(z)}{z} = \frac{z+1}{z(z-0.5)(z-1)^2} = \frac{A}{z} + \frac{B}{z-0.5} + \frac{C}{z-1} + \frac{D}{(z-1)^2}$$

We obtain the expansion coefficients as

$$A = zX_1(z)\Big|_{z=0} = \frac{z+1}{(z-0.5)(z-1)^2}\Bigg|_{z=0} = \frac{1}{(-0.5)(-1)^2} = -2$$

$$B = (z-0.5)X_1(z)\Big|_{z=0.5} = \frac{z+1}{z(z-1)^2}\Bigg|_{z=0.5} = \frac{1.5}{0.5(-0.5)^2} = 12$$

$$D = (z-1)^2 X_1(z)\Big|_{z=1} = \frac{z+1}{z(z-0.5)}\Bigg|_{z=1} = \frac{2}{1(0.5)} = 4$$

To find C, we can select any appropriate value of z and substitute in Equation 7.13.1. If we choose $z = 2$, we obtain

$$X_1(z) = \frac{X(z)}{z} = \frac{z+1}{z(z-0.5)(z-1)^2} = \frac{A}{z} + \frac{B}{z-0.5} + \frac{C}{z-1} + \frac{D}{(z-1)^2}$$

$$\frac{2+1}{2(1.5)(1)} = \frac{-2}{2} + \frac{12}{1.5} + \frac{C}{1} + \frac{4}{1}$$

$$\text{or} \quad 1 = -1 + 8 + C + 4 \rightarrow C = -10$$

Thus,

$$X(z) = -2 + \frac{12z}{z-0.5} - \frac{10z}{z-1} + \frac{4z}{(z-1)^2}$$

Taking the inverse z-transform of each term,

$$x[n] = -2\delta[n] + \{12(0.5)^n - 10 + 4n\}u[n]$$